FRACTAL GEOMETRY AND THE DISCUSSION ON FRACTAL INTERPOLATION

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Abstract

From a "strange" case-Koch curve in Euclidean space, we introduce the concept of fractal geometry, and meanwhile, give the MATLAB programs and their operational results. Since a variety of phenomena in nature have fractal characteristics, yet it is difficult to simulate the natural features with fractal characteristics in Euclidean space. Therefore, in order to solve this problem, we apply fractal interpolation to simulate natural features. Thus, based on the iterated function system (IFS), the basic principles and calculate formulas of fractal interpolation curve are given through discrete data points. Then, we realize the fractal interpolation fitting of discrete data points through the matrix calculations and graphic drawing function of MATLAB.

1. Introduction

Fractal geometry is a new branch of mathematics that is developed by Mandelbort [1], which is used to describe the phenomena and behaviours that are irregular and chaotic in nature. There are many phenomena with fractal characteristics in nature, such as the folds of the brain cortex, the trace of the lighting, the shape of the snowflake and mountains, the structure of the crystal. Their characteristics are that their local is similar or approximate to their whole. However, it is difficult for us to describe and recover with the traditional Euclid geometry. So, people use the method of interpolation to fit these irregular natural landscapes. Due to the object of interpolation is fractal, the interpolation is called fractal interpolation. Fractal interpolation function is similar to the elementary function, which has its own geometrical characteristics. So, fractal interpolation function can be expressed with a mathematical formula and be calculated quickly. Meanwhile, the main difference between them is the fractal features of fractal interpolation function, for example, it has non-integral dimensions. We can obtain the fitting results of fractal interpolation on discrete data points by using the function of strong matrix operations, graphics rendering, and data processing of MATLAB.

In [1], the French mathematician Mandelbort first proposed the idea of fractal dimension and fractal geometry and created the new term "Fractal" when he lectured at the college DE France. In [2], American mathematician Barnsley put forward the concept of fractal interpolation for the first time, he gave a new idea fitting data, which not only opened up a new research field for function approximation theory, and also provide a powerful tool for computer graphics. Now, it shows strong vitality. Lots of random generation approaches of fractal can be found in literatures [3-5]. This method can give beautiful landscape and very realistic scene in 3D Euclidean space, and meanwhile, it can provide a powerful tool for the intuitive description and graphical representation of

fractal object. The main idea of the fractal geometry is research the complex disorder chaos phenomenon of the nature. Therefore, in recent years, fractal geometry has got rapid development in theory and a real world application.

2. The "STRANGE" Phenomena in Euclidean Space [7]

In this section, we first introduce the strange phenomena of Euclidean space, such as Weierstrass function, Koch curve [8], Sierpinski triangle [8], and triadic Cantor sets [8]. In here, we mainly introduce the Koch curve in the following section. The classical Koch curve construction algorithm is very well known. An iterative function system (IFS) algorithm can be applied to generate the succession of curves that converge to the ideal fractal shape. These IFSs are used to construct certain fractals. Rather than a rigorous mathematical description that can be found elsewhere [12], only the relevant points will be outlined. Details can be found in some literatures [2, 11-12]. In the following, we will give the simple MATLAB procedure.

**Algorithm 1.** Iterative Function System (IFS) algorithm

1: $y = [1, 0]$;

2: Iterative $= t$; % t is any positive integer (the times of iteration);

3: **for** $k = 1 : t$ **do**

4:      $x = y$;

5:      $n = \text{length } (x) - 1$;

6:      **for** $m = 0 : (n - 1)$ **do**

7:        $z = (x(m + 2) - x(m + 1)) / 3$;

8:        $y(t^* m + 1) = x(m + 1)$;

9:        $y(t^* m + 2) = x(m + 1) + z$;

10:        $y(t^* m + 3) = x(t^* m + 2) + z^*((1 - \text{sqr}t(3)^* i)/2)$; % i is imaginary unit.

11:        $y(4^* m + 4) = x(m + 1) + 2^* z$;

12:      **end for**

13:      $y(4^* n + 1) = x(n + 1)$;

14:   plot(y), axis equal;

15: **end for**

The iteration results with different iteration times (one (a), two (b), three (c), and four (d)) is shown in Figure 1. The Figure 1 is also called the generator of Koch curve. From the above results, we can see that in the iteration process, the points of pictures will be more and more, in addition, the details of the Koch curves will depend on the times of iteration and the resolution ratio of display system.
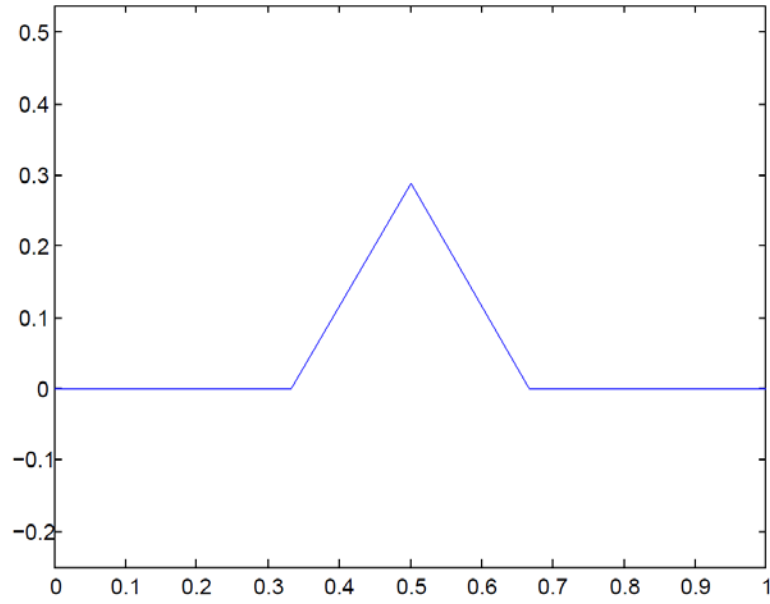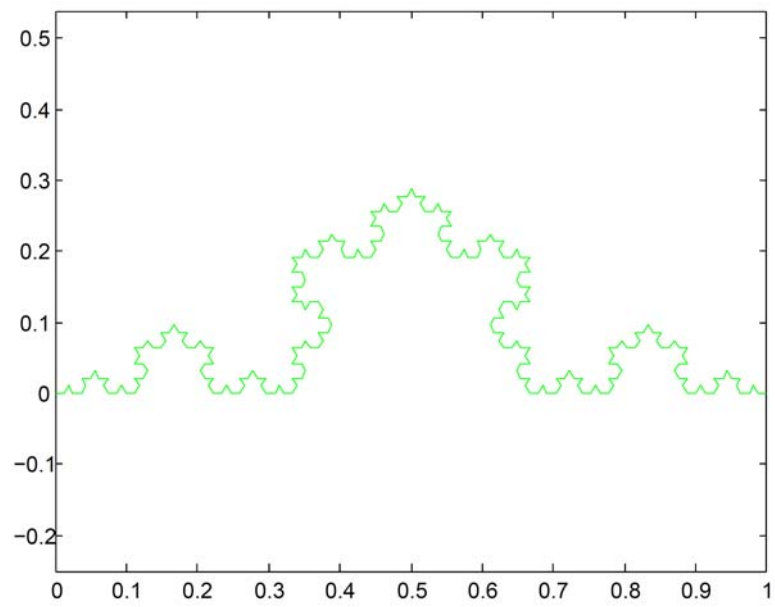
(a)
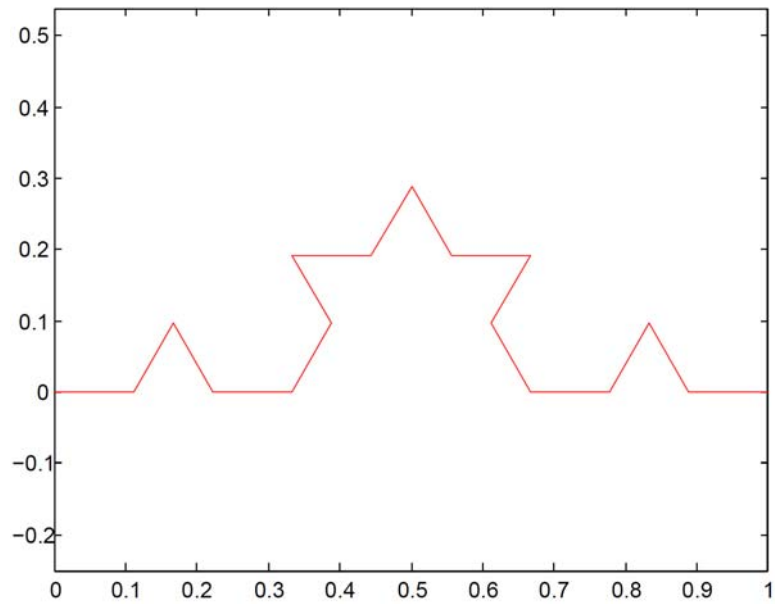
(b)
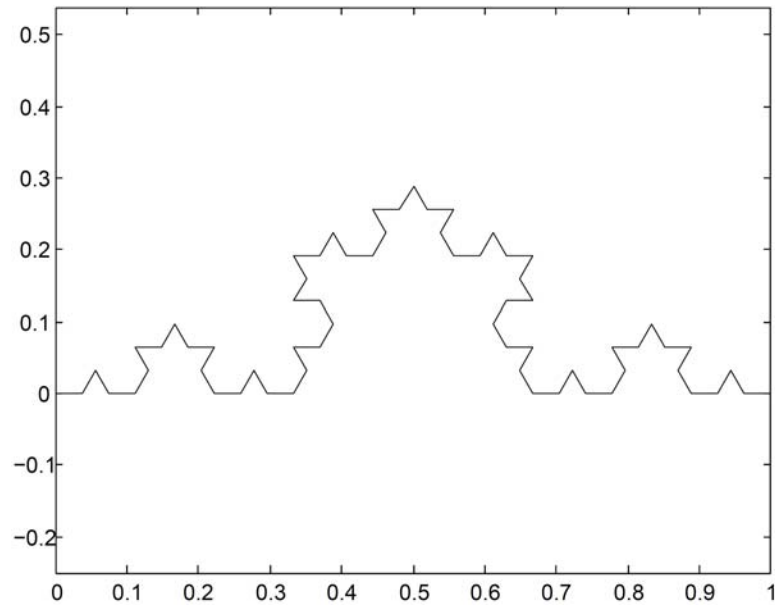
(c)

(d)

Figure 1. The iteration results with different iteration times.

3. Fractal Interpolation

In many experiments, we often face many single input and single output data (sample points) that obtained by measurement $\{(x_i, F_i) : x_{i-1} < x_i, i = 1, 2, \cdots, N\}$. In order to deal with these data, we often regard them as the points on the plane and put them in the rectangular coordinate system, then find a low-order polynomial or a basic function, and let its image through each point smoothly, continuously, and naturally as soon as possible, i.e., $F_i = f(x_i)$, $i = 0, 1, \cdots, N$. The function $f(x)$ is called interpolation function and these data points (x_i, F_i) are called interpolation points. Due to the distribution of points is more complex on the plane, we often can not guarantee that the curve will be through all the points. Therefore, there will be a fitting, that is to say, under the certain error criterion, we have $F_i \approx f(x_i)$, $i = 0, 1, \cdots, N$.

Traditional interpolation tool contains polynomial interpolation, spline interpolation, and rational function interpolation. These interpolation curves obtained by the above methods are all smooth or piecewise smooth, and their dimensions are one. Apparently, it is appropriate for us to use them to approximate or fit smooth data or smooth curves. But for many non-stationary data or non-smooth curves that exist in the actual society, such as stock cloud, seismic data, coastline, the ridge line, the outline of the cloud, electroencephalogram (egg), the traditional interpolation methods are not good approximation or fitting tools. That is to say, for the adjacent two interpolation points, we can only connect them with lines or smooth curves through traditional interpolation functions, yet we can not get the local variation characteristics between two points. For a large number of actual situation, the adjacent two sample points are not linear changeable or smooth transposition, there exist the local changeable characteristics between them. In fact, we use fractal interpolation to get the local variation characteristics between the two sample points and such that the result of interpolation is more suited to our research's actual needs. In the following section, we mainly study the fractal interpolation curve [10-11]. Although fractal interpolation has only 20 years history, it has become one of hot research fields in fractal geometry rapidly and has produced a large number of theoretical and applied results. We will introduce the principle of fractal interpolation in the following:

Fractal interpolation function is generated by a kind of special iterated function system (IFS), fractal interpolation based on iterated function system use data points to form fractal interpolation function, and regard the generated graphics as compression mapping that remains the same.

3.1. Data set

Let a point set $\{(x_i, F_i) \in \mathbb{R}^2, i = 0, 1, 2, \cdots, N\}$, where $x_0 < x_1 < \cdots < x_N$ is interpolation points. And interpolation function $f : [x_0, x_N] \to \mathbb{R}$ is a mapping, which satisfies interpolation condition $f(x_i) = F_i$, $i = 0, 1, 2, \cdots, N$.

3.2. Construction IFS

Let interpolation interval is $I = [x_0, x_N]$, two point interval is $I_i = [x_{j-1}, x_j]$. And again assume the transformation is $L_j : I \to I_j$, $j = 1, 2, \cdots, N$, where L_j is a compressed mapping, and $L_j = x_{j-1}$, $L_j(x_N) = x_j$. $F_j : K \subset I \times \mathbb{R} \to [a, b]$ is a transformation, where a, b are continuous, and $F_j(x_0, y_0) = y_{j-1}$, $F_j(x_N, y_N) = y_j$.

We define the affine transformation is $W_j(x, y) = (L_j(x), F_j(x, y))$, $j = 1, 2, \cdots, N$. And meanwhile, we can able to prove IFS with a unique attractor G. Where G is the image of continuous function $f : I \to [a, b]$, which satisfies $f(x_i) = F_i$, $i = 0, 1, \cdots, N$.

We use n to substitute the subscript j of affine transformation $W_j(x, y)$, i.e., $W_n(x, y)$ (abbreviation W_n), and then we consider iteration function system IFS $\{\mathbb{R}^2 : W_n, n = 1, 2, \cdots, N\}$, where W_n has the following forms of affine transformation:

$$W_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}, \tag{2.1}$$

and

$$W_n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ x_{n-1} \end{pmatrix}, \quad W_n\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ x_n \end{pmatrix}, \tag{2.2}$$

where

$$\begin{cases} a_n = \frac{x_n - x_{n-1}}{x_N - x_0}, \\ e_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}, \\ c_n = \frac{y_n - y_{n-1} - d_n(y_N - y_0)}{x_N - x_0}, \\ f_n = \frac{x_N y_{n-1} - x_0 y_n - d_n(x_n y_0 - x_0 y_N)}{x_N - x_0}. \end{cases} \tag{2.3}$$

Equation (2.1) is equivalent to four equations, which contains five unknown parameters. Therefore, there is a free parameter d_n, which is also called vertical scale factor, where $|d_n| < 1$, otherwise, the IFS is not convergent.

3.3. Analysis of MATLAB experiment on fractal interpolation

We consider the following example. Assume that the following interpolation points are (2,1), (3,2), (5,5), (6,4), (9,7), and (10,6). Then we draw the fractal interpolation curve under different scale factor respectively. (Let the iterated times is 2.)

From Figure 2, we can obtain the following conclusion. When the plane interpolation point set $\{(x_i, F_i)\}_{i=0}^N$ is given, affine fractal interpolation function will be determined uniquely by the vertical scale factor $\{d_n\}$, take different $\{d_n\}$, we can get different forms of affine fractal interpolation curve. Especially, $d_n = 0, n = 1, \cdots, N$, the corresponding affine fractal interpolation curve is a broken line which connects the column of interpolation points successively. Generally speaking, when the value of $|d_n|$ is greater, the corresponding fractal curve becomes more complex, oscillation is more intense. So d_n is the key factor that influences the affine fractal interpolation curve.
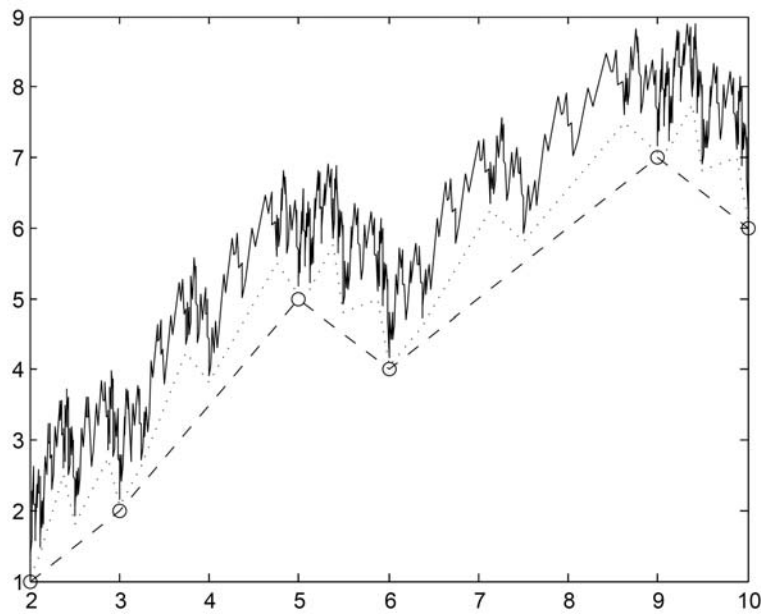
$$(d_n = 0)$$

$$(d_n = 0.1)$$

$(d_n = 0.3)$

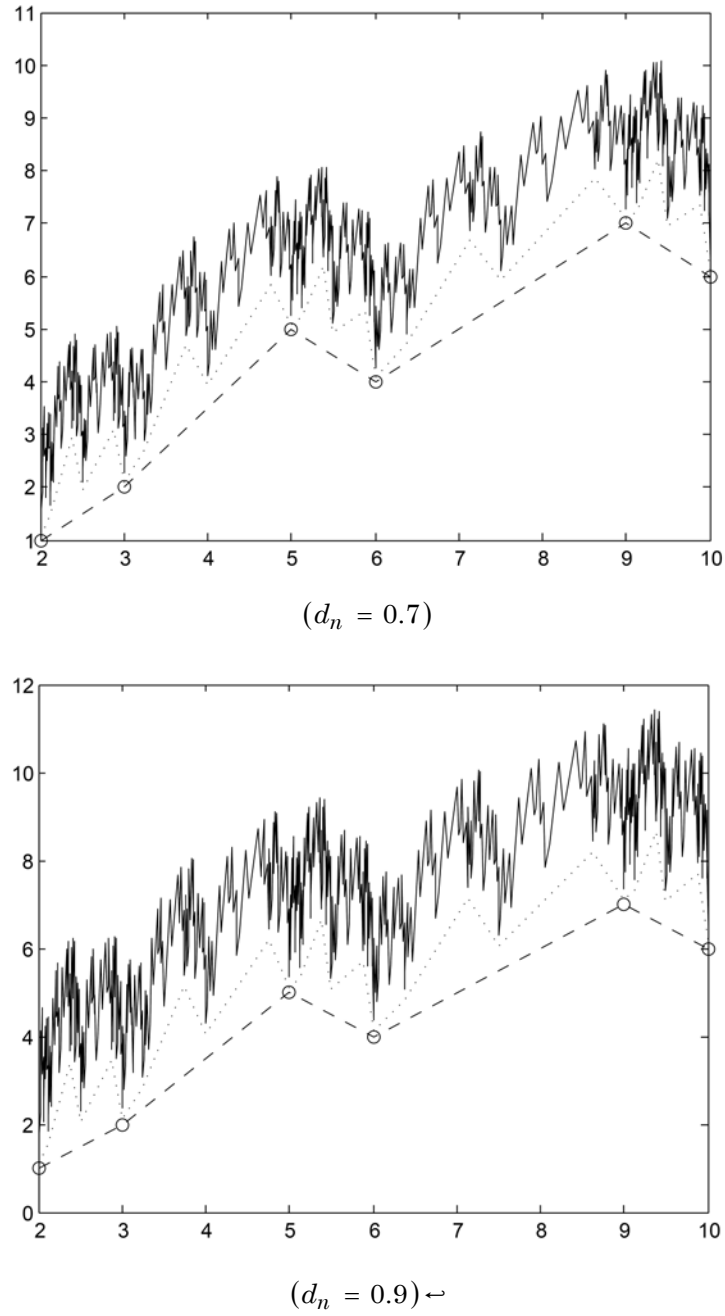$(d_n = 0.5)$

$$(d_n = 0.7)$$

$$(d_n = 0.9)$$
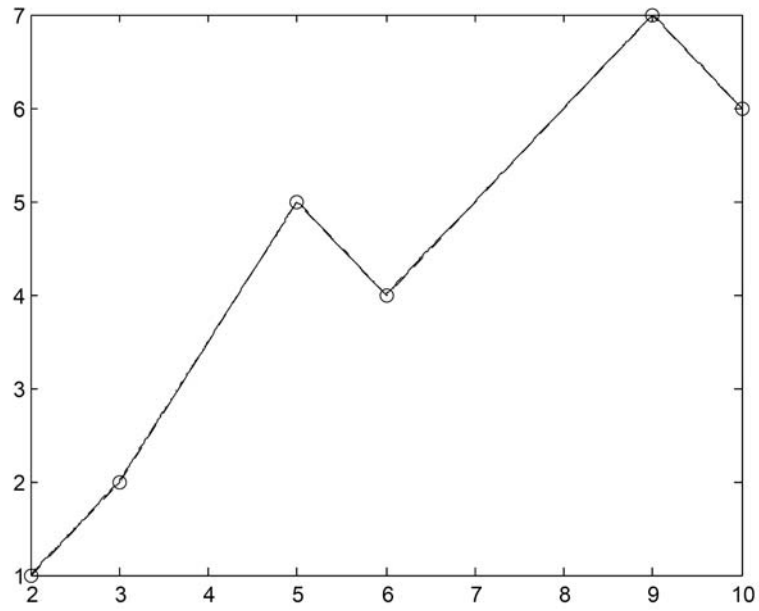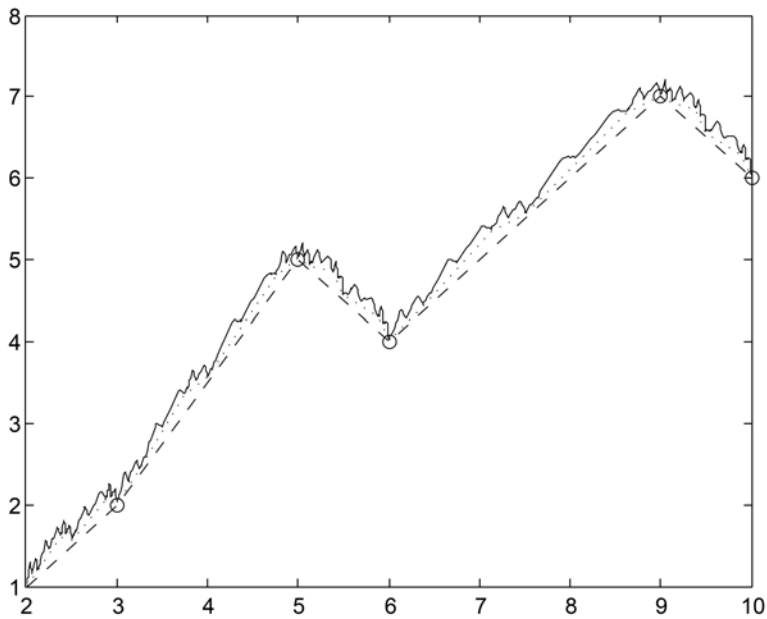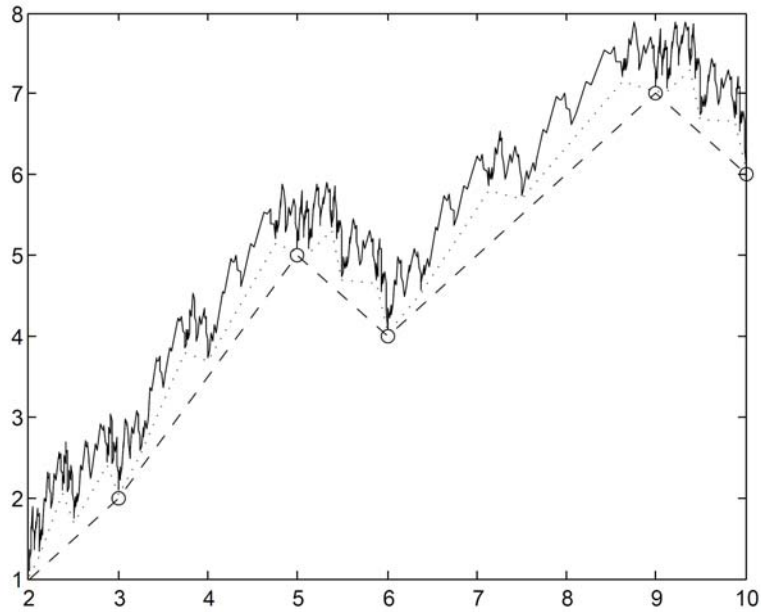
Figure 2. The fractal interpolation curves under different scale factors.

4. Conclusion

Fractal interpolation overcomes the traditional interpolation method's shortcoming that can not reflect the local characteristics between two adjacent known related points. We use fractal interpolation to simulate the natural form and can get higher precision than the traditional interpolation method. Fractal interpolation function can depict the beautiful scene in nature, moreover, it can provide a new way for fitting empirical data and give a new fitting data idea.

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References

[1]   B. B. Mandelbort, How long is the coastline of Britain? Statistical self-similarity and fractal dimension, Science 156(3775) (1967), 636-638.

[2]   M. F. Barnsley, Fractal functions and interpolation, Constructive Approximation 2(1) (1986), 303-329.

[3]   H. Ruian, H. Jiyang and X. Shugong, Computer Image and Applications of the Fractal, China Railway Press: Beijing, (1995), 61-85.

[4]   S. Wanshou and Y. Jinji, Modeling method for the earth's surface, Mini-Micro Systems 17(3) (1996), 32-36.

[5]   X. Heping, Fractals in Rock Mechanics, A Balkema Publishers: Netherlands, (1993), 70-78.

[6]   B. B. Mandelbort, The Fractal Geometry of Nature, W. H. Freeman: New York, (1982), 361-366.

[7]   Z. Congjun, W. Hongyong and S. Ping, The basis of modern analysis on mathematical economic, Higher Education Press: Beijing, (2012), 148-151.

[8]   C. Yongsheng, Make use of MATLAB to draw fractal geometry figure, Journal of Jilin Normal University 28(1) (2007), 87-89.

[9]   S. Hongquan, Fractal geometry and study on fractal interpolation, Journal of Hebei University of Technology 31(1) (2002), 56-60.

[10]   Michael F. Barnsley, Fractal Functions and Interpolations, Constructive Approximation 2: New York, 1986.

[11]   C. HuiQin, Study on fractal interpolation and its fractal dimension by graphic method solving, Jiangxi Science 28(2) (2010), 167-169.

[12]   H. O. Peitgen, H. Jurgens and D. Saupe, Chaos and Fractals, New Frontiers in Science, Springer-Verlag, New York, 1992.

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